Quantum Computation with Coherent States, Linear Interactions and Superposed Resources.

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Abstract

We show that quantum computation circuits using coherent states as the logical qubits can be constructed from very simple linear networks, conditional measurements and coherent superposition resource states.

Quantum optics has proved a fertile field for experimental tests of quantum information science, from experimental verification of Bell inequality violations [1,2] to quantum teleportation [3]. However, quantum optics was not thought to provide a practical path to efficient and scalable quantum computation, and most current efforts to achieve this have focussed on solid state implementations. This orthodoxy was challenged recently when Knill et al. [4] showed that, given single photon sources and single photon detectors, linear optics alone would suffice to implement efficient quantum computation. While this result is surprising, the complexity of the optical networks required is daunting.

In this letter we propose an efficient scheme which is elegant in its simplicity. By encoding the quantum information in multi-photon coherent states, rather than single photon states, simple optical manipulations acquire unexpected power. The required resource, which may be produced non-deterministically, is a superposition of the vacuum and a coherent state. Given this, the scheme is deterministic and requires only simple linear optics and photon counting. Qubit readout uses homodyne detection which can be highly efficient.

The idea of encoding quantum information on continuous variables of multi-photon fields has emerged recently [5] and a number of schemes have been proposed for realizing quantum computation in this way [6–8]. One drawback of these proposals is that "hard", non-linear interactions are required "in-line" of the computation. These would be very difficult to implement in practice. In contrast this proposal requires only "easy', linear in-line interactions. The hard interactions are only required for "off-line" production of resource states. A related proposal is that of Gottesman et al [9] in which superpositions of squeezed states are used to encode the qubits. There the hard interactions are only used for the initial state preparation. However, quadratic, squeezing type interactions, are required in-line along with linear interactions.

The output of a single mode, stabilised laser can be described by a coherent state, $|\alpha\rangle$ where α is a complex number which determines the average field amplitude. Coherent states are defined by unitary transformation of the vacuum [10], $|\alpha\rangle = D(\alpha)|0\rangle$, where $D(\alpha)$ is the displacement operator. Let us consider an encoding of logical qubits in coherent states with "binary pulse code modulation",

$$|0\rangle_L = |0\rangle \tag{1}$$

$$|1\rangle_L = |\alpha\rangle \tag{2}$$

where we take α to be real. The advantage of using such states is that detection is relatively easy, requiring only efficient homodyne detection.

These qubits are not exactly orthogonal, but the approximation of orthogonality is good for α even moderately large,

$$\langle\alpha|0\rangle = e^{-\alpha^2/2} \tag{3}$$

We will assume for most of this paper that $\alpha >> 1$.

In single photon optics two qubit gates, in which the state of one photon controls the state of the other, represent a formidable challenge. Surprisingly, for our coherent state encoding, a non-trivial two qubit gate can be implemented using only a single beamsplitter. Consider the beamsplitter interaction given by the unitary transformation

$$U_{BS} = \exp[i\theta(ab^\dagger + a^\dagger b)] \tag{4}$$

where a and b are the annihilation operators corresponding to two coherent state qubits $|\gamma\rangle_a$ and $|\beta\rangle_b$, with γ and β taking values of α or 0. It is well known that the output state produced by such an interaction is

$$U_{BS}|\gamma\rangle_a|\beta\rangle_b = |\cos\theta\gamma + i\sin\theta\beta\rangle_a|\cos\theta\beta + i\sin\theta\gamma\rangle_b \tag{5}$$

where $\cos^2\theta$ ($\sin^2\theta$) is the reflectivity (transmissivity) of the beamsplitter. Now consider the overlap between the output and input states. Using the relationship [10] $\langle\tau|\alpha\rangle = \exp[-1/2(|\tau|^2 + |\alpha|^2) + \tau^*\alpha]$ we find

$$\langle\gamma|_a\langle\beta|_b|\cos\theta\gamma + i\sin\theta\beta\rangle_a|\cos\theta\beta + i\sin\theta\gamma\rangle_b = \exp[-(\gamma^2+\beta^2)(1-\cos\theta) + 2i\sin\theta\gamma\beta] \tag{6}$$

Now suppose that θ is sufficiently small such that $\theta^2\alpha^2 << 1$ but that α is sufficiently large that $\theta\alpha^2$ is of order one. Physically this corresponds to an almost perfectly reflecting beamsplitter. Eq. 6 then approximately becomes

$$\langle\gamma|_a\langle\beta|_b|\cos\theta\gamma + i\sin\theta\beta\rangle_a|\cos\theta\beta + i\sin\theta\gamma\rangle_b \approx \exp[2i\theta\gamma\beta] \tag{7}$$

Eq.7 shows that the only difference between the input and output states of the beamsplitter is a phase shift proportional to the amplitudes of the input qubits, that is:

$$U_{BS}|\gamma\rangle_a|\beta\rangle_b \approx \exp[2i\theta\gamma\beta]|\gamma\rangle_a|\beta\rangle_b \tag{8}$$

If conditions are such that Eq.8 is a good approximation and we further require that $\theta\alpha^2 = \pi/2$ then this transformation produces a controlled sign shift gate. That is if either or both of the qubits are in the logical zero state ($\gamma = 0$ and/or $\beta = 0$) the transformation produces no effect on the state. However if both modes are initially in the logical one state (i.e $\gamma = \beta = \alpha$) then a sign change is produced. Such a gate is a non-trivial two qubit gate.

For universal computation we require, in addition to the two qubit gate above, the ability to do arbitrary rotations that are diagonal in the computational basis, bit-flip operations, plus the Hadamard gate [11]. The Hadamard gate cannot be implemented unitarily with linear optics. However, we will show shortly that, provided the necessary quantum resource is possessed, it can be implemented using only linear optics and conditional measurements.

First let us consider some single qubit transformations that can be achieved with just linear optics. A *bit flip* gate flips the state of the system from a logical zero to a logical one, or vice versa and is equivalent to the pauli $\sigma_x \equiv X$ matrix, in the computational basis. The bit flip transformation operator, X, is equivalent to a displacement of $-\alpha$ followed by a π phase shift of the coherent amplitude:

$$X = U(\pi)D(-\alpha) \tag{9}$$

where $U(\pi) = exp[i\pi a^\dagger a]$ is physically just a half-wavelength delay, whilst a displacement can be implemented by mixing a very strong coherent field with the qubit on a highly reflective beamsplitter [5].

The *phase rotation* gate produces a rotation that is diagonal in the computational basis, $R_\phi(\mu|0\rangle_L + \nu|1\rangle_L) = \mu|0\rangle_L + e^{i\phi}\nu|1\rangle_L$. It can be implemented, to a good approximation, by imposing a small phase shift on the qubit. Using arguments similar to those leading to Eq.8 we find

$$\begin{aligned} U(\epsilon)|\alpha\rangle &= e^{i\epsilon a^\dagger a}|\alpha\rangle \\ &\approx e^{i\epsilon\alpha^2}|\alpha\rangle = R_\phi|\alpha\rangle \end{aligned} \tag{10}$$

with $\phi = \epsilon\alpha^2$. We have assumed ϵ scales as $1/\alpha^2$.

In addition to these gates, we require a Hadamard gate in order to achieve an arbitrary qubit rotation. The Hadamard gate, $\mathcal{H}$, induces the following transformations on the logical states:

$$\begin{aligned} \mathcal{H}|0\rangle_L &= |0\rangle_L + |1\rangle_L = |0\rangle + |\alpha\rangle \\ \mathcal{H}|1\rangle_L &= |0\rangle_L - |1\rangle_L = |0\rangle - |\alpha\rangle \end{aligned} \tag{11}$$

The outputs are a superposition of two widely separated coherent states, commonly known as "cat" states. Such states are highly non classical and for unitary generation require a Kerr nonlinearity for which the Hamiltonian is proportional to $(a^\dagger a)^2$. Such interactions are typically very weak and do not have sufficient strength to produce the required superposition states. However we are not restricted to unitary transformations. A number of schemes have been suggested which can produce parity cat states non-deterministically [12,13] and some experimental progress has been made in their production [14–16]. In all these schemes it is necessary to distinguish between a photon (or phonon) number of n and $n \pm 1$. If cat states could be used as a resource to deterministically implement the Hadamard gate then these types of schemes would be sufficient for our purposes. We will now show this is true.

A Hadamard gate can be implemented using the two qubit BS gate discussed in the previous section with one of the inputs being the arbitrary state we wish to transform and the second input being a known cat state. One of the outputs of the gate is measured in the "cat basis" (see below) and, depending on the result, a bit flip operation may be required. This is a specific example of quantum gate implementation via measurement. A general discussion of such techniques can be found in Reference [11].

A possible arrangement is shown in Fig.1. Suppose the state we wish to transform, in the arbitrary state $\mu|0\rangle + \nu|\alpha\rangle$, is inserted into port 1 of the BS gate whilst a resource cat state $1/\sqrt{2}(|0\rangle + |\alpha\rangle)$ is inserted into port 2. The output state of the gate is

$$\frac{\mu}{\sqrt{2}}(|0\rangle_1|0\rangle_2 + |0\rangle_1|\alpha\rangle_2) + \frac{\nu}{\sqrt{2}}(|\alpha\rangle_1|0\rangle_2 - |\alpha\rangle_1|\alpha\rangle_2) \tag{12}$$

Now suppose we make a measurement on output port 1 which returns a dichotomic result telling us whether we have the same cat state as we inserted or the (near) orthogonal state $1/\sqrt{2}(|0\rangle - |\alpha\rangle)$. If the result is the same cat state then the state of output port 2 is projected into

$$\frac{1}{2}(\mu + \nu)|0\rangle + \frac{1}{2}(\mu - \nu)|\alpha\rangle \tag{13}$$

This is the required Hadamard transformation. On the other hand if the opposite cat is measured at the output as was inserted then the projected output state is

$$\frac{1}{2}(\mu - \nu)|0\rangle + \frac{1}{2}(\mu + \nu)|\alpha\rangle \tag{14}$$

But the state of Eq.14 only differs from that of Eq.13 by a bit flip operation. Thus the final step of the gate is to implement (if necessary) a bit flip on the output port.

A cat basis measurement can be implemented in the following way. First we displace by $-\alpha/2$. This transforms our "0", "α" superposition into "$\alpha/2$", "$-\alpha/2$" superposition:

$$D(-\alpha/2)1/\sqrt{2}(|0\rangle \pm |\alpha\rangle) = 1/\sqrt{2}(|-\alpha/2\rangle \pm |\alpha/2\rangle) \tag{15}$$

These new states are parity eigenstates. Thus if photon number is measured then an even result indicates detection of the state $1/\sqrt{2}(|\alpha/2\rangle + |-\alpha/2\rangle)$ and therefore $1/\sqrt{2}(|0\rangle + |\alpha\rangle)$ whilst similarly an odd result indicates detection of $1/\sqrt{2}(|0\rangle - |\alpha\rangle)$ as can be confirmed by direct calculation. The cats could also be distinguished by homodyne detection looking at the imaginary quadrature [17]. This latter technique would give inconclusive results some of the time but may be useful for initial experimental demonstrations.

The control not gate (CNOT) is ubiquitous in quantum processing tasks. It is also the simplest two-qubit gate whose operation can easily be experimentally verified in the computational basis. A CNOT gate will flip the state of one of the input qubits, the "target", only if the other qubit, the "control", is in the logical one state. If the control is in the logical zero state the target is unchanged. A CNOT gate can be implemented as shown in Fig.2 by first applying a Hadamard gate to the target state followed by the BS gate applied to the control and target. Finally another Hadamard gate is applied to the target. For arbitrary control and target input qubits we find:

$$\mathcal{H}_t U_{BS} \mathcal{H}_t (\mu|0\rangle + \nu|\alpha\rangle)_c (\gamma|0\rangle + \tau|\alpha\rangle)_t = \mu\gamma|0\rangle|0\rangle + \mu\tau|0\rangle|\alpha\rangle + \nu\tau|\alpha\rangle|0\rangle + \nu\gamma|\alpha\rangle|\alpha\rangle \tag{16}$$

which displays CNOT logic. The result of Eq.16 assumes $\alpha >> 1$. To evaluate just how large α needs to be we use the exact expression for the BS gate, as given in Eq.5, to calculate the output state of the CNOT. We assume ideal bit flip operations and cat state preparation. Our figure of merit is the average fidelity between the exact output and the ideal output, as given by Eq.16.

The results are shown in Fig.3. In Fig.3(a) the average fidelity is plotted as a function of α. Fidelities of 0.9 and above require $\alpha > 10$. Such signal sizes, although commonplace in the computational basis would be challenging to produce and control in the superposition basis and the required technology is probably some years away. On the other hand in Fig.3(b) a renormalised average fidelity is plotted. This is obtained by normalising the fidelity of getting the correct output state against the sum of the fidelities for all the possible output states in the computational subensemble. If there was no movement of states out of this subensemble one would expect the two plots to be identical. The fact that the renormalised fidelities remain high for much lower values of α shows that qubit leakage is the major reason for the decreasing fidelities at moderate levels of α in Fig.3(a). This in turn suggests that experimental demonstrations, albeit with low efficiency, may be possible for α's as small as 3.

The major sources of error in our scheme are expected to be, in order of increasing significance: (i) errors due to non orthogonal code states, (ii) errors due to failure of the two qubit gate condition ($\theta^2\alpha^2 << 1$), (iii) erroneous identification of the input cat resource, (iv) photon loss, and (v) errors due to random optical phase shifts. The first source of error becomes negligible for $\alpha > 3$ (see Eq.3). Figure 3(a) shows that the second source of errors is small for $\alpha > 20$. The third source is equivalent to a small rotation error in the code space; the fourth source causes a collapse to the one logical state, while the final source is a phase error. It can be shown [17] that good quantum error correction codes are available to correct these errors and further that error correction can be implemented in a fault tolerant fashion.

In this letter we have presented a quantum computation scheme based on encoding qubits as vacuum and coherent states, and their superposition. The optical networks required are simple and compatible with current optical communication networks. As well as the long term goal of quantum computation, applications in quantum communication protocols seem likely. Although the coherent amplitudes needed for scalable computation are quite large our results indicate that experimental demonstrations with modest amplitudes should be possible.

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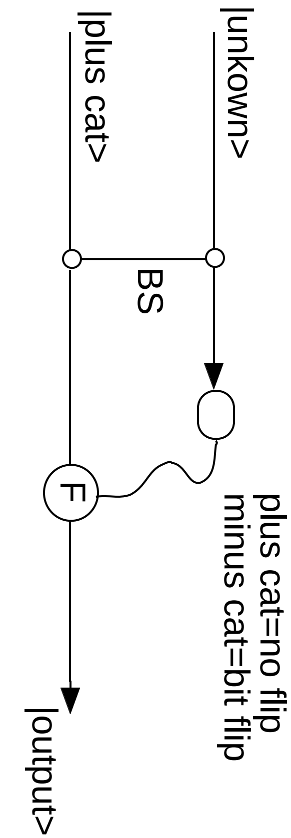
FIGURES

FIG. 1. Schematic of Hadamard gate based on the two qubit beamsplitter gate (BS) and conditional implementation of the bit flip gate (F). If an even number of photons is counted then the output is in the desired state. If an odd number of photons is counted a bit flip operation is needed to place the output in the correct state.

FIG. 2. Schematic of CNOT gate based on two qubit beamsplitter gates (BS) and conditional implementation of bit flip gates (F).

FIG. 3. Performance of the CNOT gate as a function of the magnitude of α. In (a) the average fidelity is plotted against α. The high fidelities at very low values of α are an artefact of the non-orthogonality of small α states. In (b) the fidelities are renormalised against the total fidelity for a result within the computational subensemble of states.

Ralph_fig1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BS
F
plus cat=no flip
minus cat=bit flip
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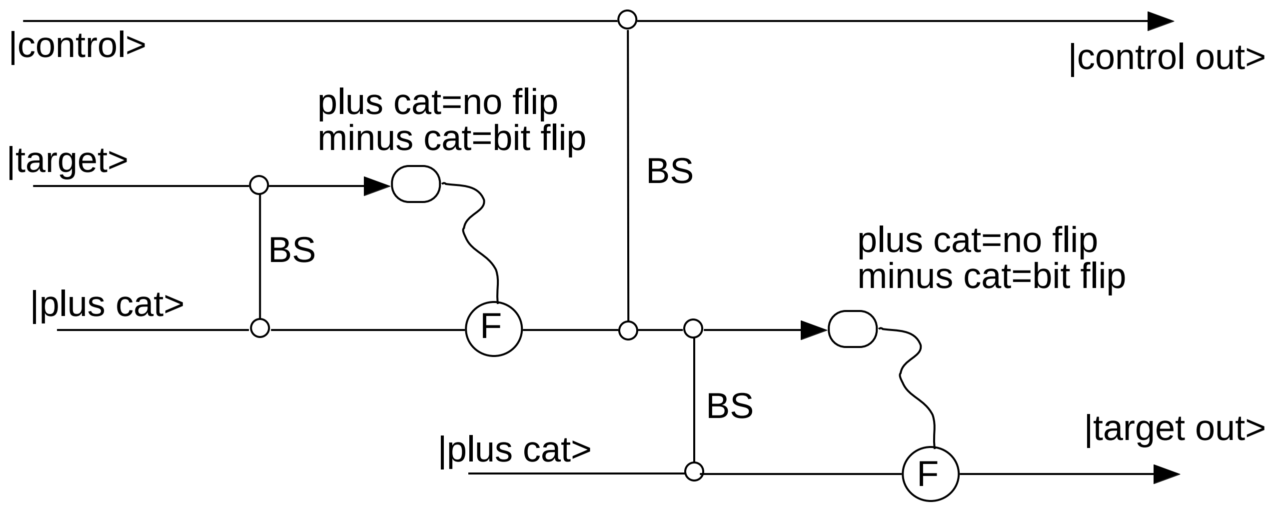
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minus cat=bit flip
BS
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BS
|plus cat>
F
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BS
|plus cat>
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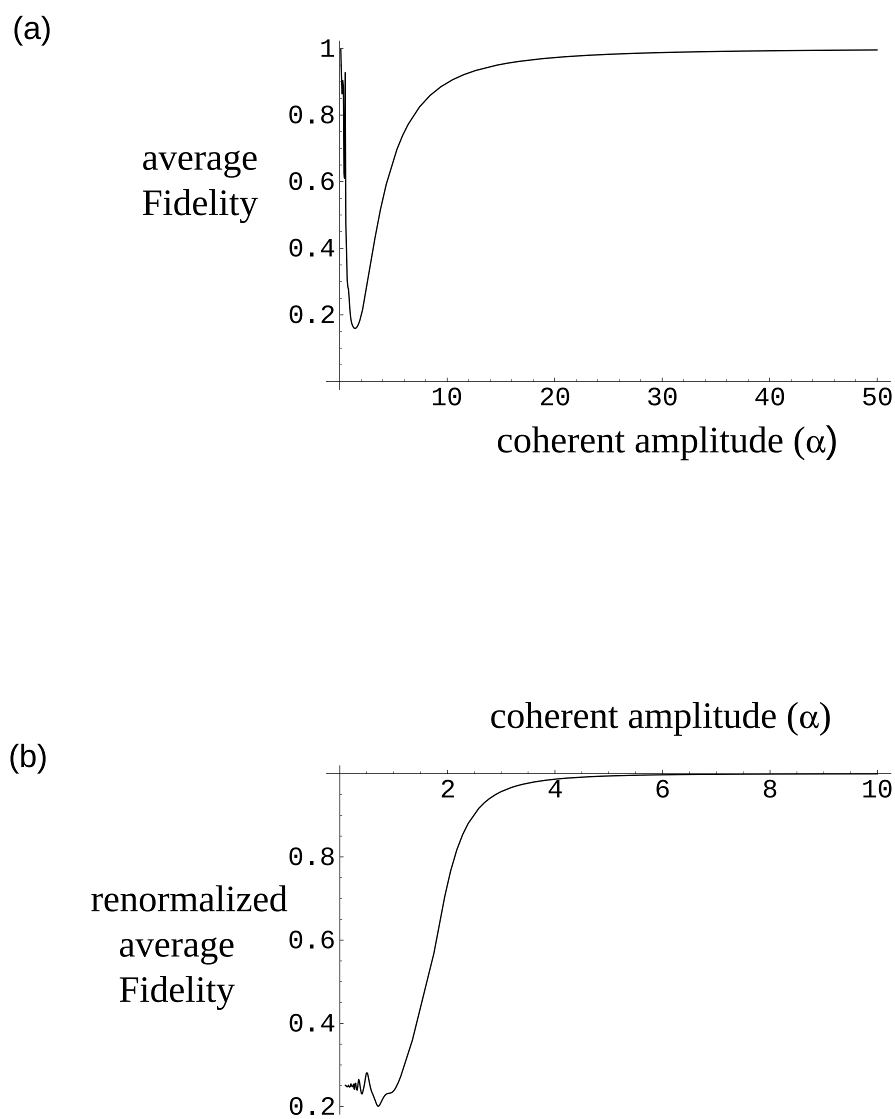
F

Ralph_fig2


(a)
1
0.8
0.6
0.4
0.2
average
Fidelity
10
20
30
40
50
coherent amplitude (α)
(b)
coherent amplitude (α)
2
4
6
8
10
0.8
0.6
0.4
0.2
renormalized
average
Fidelity

Ralph_fig3